

A note about stratified group actions

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Abstract

In this short note we aim at proving one property of constructible sheaves on quotient stratified topological stacks: namely, the fact that constructible sheaves do not see the difference between quotienting by a group G or by a quotient (G/H) where H is contractible. We also prove a relative version of this result.

Recall the definition of conically stratified topological space, from [Lur17, Definition A.5.5], and of the ∞ -category of constructible sheaves with values in spaces

$$\mathrm{Cons}(Y, s; \mathcal{S})$$

([Lur17, Definition A.5.2]). For the purposes of this note, any stratified space for which the Exodromy theorem [Lur17, Theorem A.9.3] works will be enough. We denote the category of conically stratified spaces, locally of singular shape, and with stratifying poset satisfying the ascending chain condition, by

$$\mathrm{StrTop}_{\mathrm{con}}.$$

Recall 1. Recall from [Noc20] the notion of smooth stratified submersion and stratified homotopy equivalence between stratified spaces.

Recall also that, thanks to the Exodromy theorem, given any stratified map $f : (Y, s) \rightarrow (Z, t)$, the pullback functor

$$\mathrm{Cons}(Z, t; \mathcal{S}) \rightarrow \mathrm{Cons}(Y, s; \mathcal{S})$$

admits a left adjoint, which we denote by $f_{\#}^c$.

Recall now that by [Noc20, Eq. (B.11)], the smooth base change formula holds, i.e. when

$$\begin{array}{ccc} (X', t') & \xrightarrow{\alpha_0} & (X, t) \\ \downarrow \beta_1 & & \downarrow \beta_0 \\ (Y', s') & \xrightarrow{\alpha_1} & (Y, s) \end{array}$$

is a diagram of stratified spaces, and β_0, β_1 are smooth stratified submersions, then

$$\beta_{1,\#}^c \alpha_0^* \rightarrow \alpha_1^* \beta_{0,\#}^c \tag{1}$$

is an equivalence of functors $\mathrm{Cons}(X, t; \mathcal{S}) \rightarrow \mathrm{Cons}(Y', s'; \mathcal{S})$.

Recall finally that by [Noc20, Proposition B.59], the functor

$$\mathrm{Cons}(-; \mathcal{S}) : \mathrm{StrTop}_{\mathrm{con}}^{\mathrm{op}} \rightarrow \mathcal{P}^{\mathrm{R}}$$

sends stratified homotopy equivalences to equivalences of ∞ -categories.

Recall 2. Recall from [Noc20] the definition of the category $\text{StrTStk}_{\text{con}}$ of conically stratified topological stacks. The functor $\text{Cons}(-; \mathcal{S})$ extends to $\text{StrTStk}_{\text{con}}^{\text{op}}$ by right Kan extension.

A smooth stratified submersion (resp. a stratified homotopy equivalence) is map of stratified topological stacks which can be written as a colimit of maps between representables which are smooth stratified submersions (resp. stratified homotopy equivalences).

We are interested in a special class of stratified topological stacks. Let $(S, s), (G, w), (Y, t) \in \text{StrTop}_{\text{con}}$, $(G, w) \rightarrow (S, s)$ a smooth stratified submersion exhibiting G as a relative unstratified topological group over S (i.e. the stratification of G is induced by s , and each fiber is a topological group). Let $(Y, t) \rightarrow (S, s)$ be a stratified map, and suppose there is a stratified action of G on Y relative over S .

Then we define the quotient stack

$$Y/G = \text{colim} \left[\dots \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} (G \times G \times Y, s_2) \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} (G \times Y, s_1) \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} (Y, s) \right].$$

where s_i is the stratification on $\overbrace{G \times \dots \times G}^i \times Y$ which is trivial on the group factors and s on the last factor, and the colimit is taken in the category of topological stacks. As usual, in the left direction are induced by the identity element of G in various ways, and maps in the right direction are induced by combinations of the action and the projections.

The stratification coming with this definition is denoted again by s .

By definition, we have that

$$\text{Cons}(Y/G, s; \mathcal{S})$$

is the limit of the diagram

$$\dots \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \text{Cons}(G \times G \times Y, s_2; \mathcal{S}) \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \text{Cons}(G \times Y, s_1; \mathcal{S}) \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \text{Cons}(Y, s; \mathcal{S}).$$

Note that smooth base change and stratified homotopy invariance follow formally from Recall 1 by taking limits.

Lemma 3. Let G be a topological group which is also topological manifold, $H < G$ a closed normal subgroup with the same property and furthermore contractible, and (Y, s) a conically stratified topological space with a G -action such that the restriction to H is trivial. Then the pullback map

$$\text{Cons}(Y/(G/H), s) \rightarrow \text{Cons}(Y/G, s),$$

where s stays for the stratification inherited by the quotients on both sides, is an equivalence.

Proof. We have a diagram

$$\begin{array}{ccccccc} & & & & \dots & \longrightarrow & Y & \longrightarrow & Y & \longrightarrow & Y \\ & & & & & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \dots & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & H \times H \times Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & H \times Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & G \times G \times Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & G \times X & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & G/H \times G/H \times Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & G/H \times Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & Y \end{array}$$

where all squares of the form

$$\begin{array}{ccc}
 & & X \\
 & \nearrow & \\
 H^n \times Y & & \\
 \downarrow & & \\
 G^n \times Y & & \\
 \downarrow & \nwarrow & \\
 (G/H)^n \times Y & &
 \end{array}$$

are pullback squares. Note that, at each term, the map between the (H, Y) row and the $(*, Y)$ row is a stratified homotopy equivalence, and the homotopies witnessing this can be chosen to be compatible with the rest of the diagram, since the action of H on Y is trivial. Also, $G \rightarrow G/H$ is an (unstratified) Serre fibration of trivially stratified spaces¹, hence also the map between the (G, Y) -row and the $(G/H, Y)$ -row is a stratified homotopy equivalence. Therefore, it induces a stratified homotopy equivalence of stacks on the colimits of the two rows, and therefore an equivalence at the level of constructible sheaves by Recall 2. $\square$

Lemma 4. *Let (S, s) be a stratified topological space, $G \rightarrow S$ a smooth stratified submersion exhibiting G as a relative topological group over S , $H \subset G$ a relative normal subgroup (with the inherited stratification) such that, for each stratum W of S , the map*

$$G \times_S W \rightarrow G/H \times_S W$$

is a stratified homotopy equivalence. Let $(Y, t) \rightarrow (S, s)$ be a stratified map of stratified topological spaces, and suppose there is a stratified action of G on Y relative over S , which restricts to the trivial action of H . Then the pullback map

$$\text{Cons}(Y/(G/H), s) \rightarrow \text{Cons}(Y/G, s),$$

where s stays for the stratification inherited by the quotients on both sides, is an equivalence.

Proof. As in Lemma 3, we reduce to checking that the maps

$$\pi_n^* : \text{Cons}(\underbrace{(G/H) \times_S \cdots \times_S (G/H)}_n \times_S Y) \rightarrow \text{Cons}(\underbrace{G \times_S \cdots \times_S G}_n \times_S Y),$$

given by pullback along the projection $\pi_n : \underbrace{G \times_S \cdots \times_S G}_n \times_S Y \rightarrow \underbrace{(G/H) \times_S \cdots \times_S (G/H)}_n \times_S Y$, are equivalences for every n . By Recall 1 there is an adjunction $\pi_{n,\#}^c \dashv \pi_n^*$, hence it suffices to check that for each

$$\begin{aligned}
 \mathcal{F} &\in \text{Cons}(\underbrace{(G/H) \times_S \cdots \times_S (G/H)}_n \times_S Y), \\
 \mathcal{G} &\in \text{Cons}(\underbrace{G \times_S \cdots \times_S G}_n \times_S Y)
 \end{aligned}$$

the unit and counit maps

$$\begin{aligned}
 \mathcal{F} &\rightarrow \pi_n^* \pi_{n,\#}^c \mathcal{F} \\
 \pi_{n,\#}^c \pi_n^* \mathcal{G} &\rightarrow \mathcal{G}
 \end{aligned}$$

¹Indeed, the multiplication action of H on G is free and continuous, hence the quotient map is a Serre fibration. It is a trivial Serre fibration because the fiber is contractible.

are equivalences. By the recollement principle [BGH20, Lemma 8.2.10] it suffices to check the statements after applying pullback to the strata of S to both maps. By smooth base change (1), pullback to strata commutes with $\pi_{\sharp}^c \pi^*$, and hence we reduce to proving the statement for $\mathcal{F} \in \text{Cons}(W \times_S ((G/H) \times_S \cdots \times_S (G/H)))$, $\mathcal{G} \in \text{Cons}(W \times_S (G \times_S \cdots \times_S G))$, and π replaced by $\pi|_{W \times_S (G \times_S \cdots \times_S G)}$, for every stratum W of S .

In that setting, however, the map $G \rightarrow G/H$ restricts to a stratified homotopy equivalence by hypothesis, hence the claim. $\square$

References

- [BGH20] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy. https://www.maths.ed.ac.uk/~cbarwick/papers/exodromy_book.pdf, 2020.
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